Fourier Transform

Syllabus:-

Definition, Fourier integral, Fourier transform, inverse transform, Fourier transform of derivatives, convolution (mathematical statement only), Parseval's theorem (statement only), Applications

Fourier series

Any periodic function $f(t)$ having period T satisfying Dirichlet condition can be expressed by the following series

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\omega t + b_n \sin n\omega t)$$

Where

$$a_0 = \frac{1}{T}\int_0^T f(t)dt \;,\quad a_n = \frac{2}{T}\int_0^T f(t)\cos n\omega t \; dt$$

$$\& \quad b_n = \frac{2}{T}\int_0^T f(t)\sin n\omega t \; dt$$

Dirichlet conditions

The Dirichlet condition for periodic function are;

i) The function must be periodic.
ii) The function has finite number of discontinuities in each period
iii) The function has finite number of maxima and minima in each period.
iv) The function must converge over any period. That is

$$\int_0^T |f(t)|\, dt \quad \text{is finite}$$

For periodic function $f(x)$ having period L satisfying Dirichlet condition can be expressed by the following series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{2\pi nx}{L}\right) + b_n \sin\left(\frac{2\pi nx}{L}\right)\right)$$

Where

$$a_0 = \frac{1}{L}\int_{x_0}^{x_0+L} f(x)dt \;,$$

$$a_n = \frac{2}{L}\int_{x_0}^{x_0+L} f(x)\cos\left(\frac{2\pi nx}{L}\right) dx$$

$$\& \quad b_n = \frac{2}{L}\int_{x_0}^{x_0+L} f(x)\sin\left(\frac{2\pi nx}{L}\right) dx$$

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Fourier integral

For periodic function $f(x)$ defined in interval$[-l,\ l]$ is represented in integral form as

$$f(x) = \frac{1}{\pi}\int_0^\infty d\omega \int_0^\infty f(t)\cos\omega(t-x)\,dt$$

Is called as **Fourier integral**.

If $\cos\omega(t-x)$ is replaced in complex form,

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x} d\omega \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt$$

is called **Fourier complex integral**.

Fourier transform

The complex form of Fourier integral

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x} d\omega \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt$$

This may be expressed as ,

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\omega x}\, F(\omega) d\omega \quad \dots\dots..(1)$$

Where $F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\omega t}\, f(t) dt \quad \dots\dots\dots(2)$

The function $F(\omega)$ is called Fourier transform of $f(t)$ and $f(t)$ is called as inverse transform of $F(\omega)$.

Eq^n (1) is transform of frequency function $F(\omega)$ into position function $f(x)$.

Eq^n (2) is transform of time function $f(t)$ into frequency function $F(\omega)$.

Fourier Sine and Cosine transform

1) **Fourier Cosine transform**

If $f(t)$ is even function ie. $f(-t) = f(t)$

Then Fourier cosine transform is given by

$$F_C(\omega) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\cos\omega t\ dt$$

And inverse Fourier cosine transform is given by

$$f(t) = \sqrt{\frac{2}{\pi}}\int_0^\infty F_C(\omega)\cos\omega t\ d\omega$$

2) **Fourier Sine transform**

If $f(t)$ is odd function ie. $f(-t) = -f(t)$

Then Fourier sine transform is given by

$$F_S(\omega) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\sin\omega t\ dt$$

And inverse Fourier sine transform is given by

$$f(t) = \sqrt{\frac{2}{\pi}}\int_0^\infty F_S(\omega)\sin\omega t\ d\omega$$

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Properties of Fourier transform

i) **Linear property** :- If function $F(\omega)$ is called Fourier transform of $f(t)$ &
If $f(t) = a_1 f_1(t) + a_2 f_2(t) + ...+ a_n f_n(t) = \sum a_n f_n(t)$,

then the Fourier transform of $f(t)$ is given by

$$F(\omega) = a_1 F_1(\omega) + a_2 F_2(\omega) + ...+ a_n F_n(\omega) = \sum a_n F_n(\omega)$$

ii) **Change of scale property**:- If function $F(\omega)$ is called Fourier transform of $f(t)$, then Fourier transform of $f(at)$ is $\frac{1}{a} F\left(\frac{\omega}{a}\right)$

iii) **Shifting property**:- If function $F(\omega)$ is called Fourier transform of $f(t)$, then Fourier transform of $f(t-a)$ is $e^{i\omega a} F(\omega)$

iv) **Conjugate property** :- If function $F(\omega)$ is called Fourier transform of $f(t)$, then Fourier transform of complex conjugate of $f(t)$ (ie. $f^*(t)$) is . $F^*(-\omega)$ (complex conjugate of $F(-\omega)$)

ie. $F.T\ [f^*(t)] =. F^*(-\omega)$

v) **Modulation property** :- If function $F(\omega)$ is called Fourier transform of $f(t)$, then Fourier transform of $f(t) \cos at$ is $\frac{1}{2}[F(\omega + a) + F(\omega - a)]$

vi) **Convolution property (Convolution Theorem)** :-

The convolution of two function $f(t)$ and $g(t)$ over the interval $(-\infty, \infty)$ is defined as

$$f(t) * g(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u)\, g(t-u)\, dt$$

Convolution theorem states that "If $F(\omega)$ and $G(\omega)$ be the Fourier transforms of $f(t)$ & $g(t)$ respectively, then the Fourier transform of the convolution of $f(t)$ & $g(t)$ is the product of their Fourier transform". That is

$$F.T\ [f(t) * g(t)] = F.T\ [f(t)] * F.T\ [g(t)] = F(\omega) * G(\omega)$$

Parsvel's theorem

If $F(\omega)$ and $G(\omega)$ be the Fourier transforms of $f(t)$ & $g(t)$ respectively then,

i) $$\int_{-\infty}^{\infty} f(t)\, g^*(t)\, dt = \int_{-\infty}^{\infty} F(\omega)\, G^*(\omega)\, dt$$

Where $G^*(\omega)$ is Fourier transform of $g^*(t)$ and $g^*(t)$ is complex conjugate of $g(t)$.

ii) $$\int_{-\infty}^{\infty} [f(t)^2] dt = \int_{-\infty}^{\infty} [F(\omega)^2] d\omega$$

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Derivative of Fourier transform

$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega t} f(t) dt$

Differentiating both side with respect to ω

$\frac{dF(\omega)}{d\omega} = \frac{1}{\sqrt{2\pi}} \frac{d}{d\omega} \int_{-\infty}^{\infty} e^{-i\omega t} f(t) dt$

$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\delta y}{\delta \omega} \left[e^{-i\omega t} f(t)\right] dt$

$= -i \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} t\, f(t)\, e^{-i\omega t}\ dt$

$= -i\, F.T\, [t\, f(t)]$

If $F(\omega)$ differentiated n times then

$\frac{d^n(\omega)}{d\omega^n} == (-i\,)^n F.T\, [t^n\, f(t)]$

Fourier transform of derivative

Let $F_1(\omega)$ be the Fourier transform of first derivative of function $f(t)$ then

$$F_1(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{df(t)}{dt} e^{-i\omega t}\, dt$$

Integrating by parts, we get

$$F_1(\omega) = \frac{1}{\sqrt{2\pi}} \left[f(t) e^{-i\omega t}\right]_{-\infty}^{\infty} + \frac{i\omega}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t}\, dt$$

$$F_1(\omega) = \frac{1}{\sqrt{2\pi}} \left[f(t) e^{-i\omega t}\right]_{-\infty}^{\infty} + i\omega F(\omega)$$

As $t \to \infty$, $e^{-i\omega t} \to 0$ & $t \to -\infty$, $e^{-i\omega t} \to \infty$. Therefore for existence of limit , the function $f(t)$ should be much faster decrease to zero as $t \to \infty$ than $t \to -\infty$. Then in such case

$\frac{1}{\sqrt{2\pi}} \left[f(t) e^{-i\omega t}\right]_{-\infty}^{\infty} = 0$

$\therefore F_1(\omega) = i\omega F(\omega)$

Similarly $F_n(\omega)$ be the Fourier transform of n^th derivative of function $f(t)$ then

$F_n(\omega) = (i\omega)^n F(\omega)$

Application of Fourier transform

1) Evaluation of integrals

Using Fourier transform certain integrals may be obtained.

Ex. Find the Fourier sine and cosine transform of $f(t) = e^{-pt}, p > 0$. Hence evaluate $\int_0^{\infty} \frac{\cos \omega t}{p^2+\omega^2} d\omega$ and $\int_0^{\infty} \frac{\omega \sin \omega t}{p^2+\omega^2} d\omega$

Solution:- Given $f(t) = e^{-pt}, p > 0$

By taking cosine transform of $f(t)$

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$$F_C(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t)\cos\omega t\ dt$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-pt}\cos\omega t\ dt$$

$$= \sqrt{\frac{2}{\pi}} \frac{P}{p^2+\omega^2} \qquad \dots .(1)$$

By taking inverse cosine transform $F_C(\omega)$

$$f(t) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_C(\omega)\cos\omega t\ d\omega$$

$$f(t) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \frac{P}{p^2+\omega^2} \quad \cos\omega t\ d\omega$$

$$f(t) = \frac{2P}{\pi} \int_0^\infty \frac{\cos\omega t}{p^2+\omega^2} \quad d\omega$$

$$\therefore \int_0^\infty \frac{\cos\omega t}{p^2+\omega^2} \quad d\omega = \frac{\pi}{2P} f(t) = \frac{\pi}{2P} e^{-pt} \dots \dots (2)$$

By taking sine transform

$$F_S(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t)\sin\omega t\ dt$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-pt}\sin\omega t\ dt$$

$$= \sqrt{\frac{2}{\pi}} \frac{\omega}{p^2+\omega^2} \qquad \dots .(3)$$

By taking inverse sine transform $F_C(\omega)$

$$f(t) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_S(\omega)\sin\omega t\ d\omega$$

$$f(t) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \frac{\omega}{p^2+\omega^2} \quad \sin\omega t\ d\omega$$

$$f(t) = \frac{2}{\pi} \int_0^\infty \frac{\omega\sin\omega t}{p^2+\omega^2} \quad d\omega$$

$$\therefore \int_0^\infty \frac{\omega\sin\omega t}{p^2+\omega^2} \quad d\omega = \frac{\pi}{2} f(t) = \frac{\pi}{2} e^{-pt} \dots \dots (4)$$

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2) Fourier transform to solve differential equation

Ex. To illustrate the application of Fourier transform in solving differential equations, we consider the motion of a damped harmonic oscillator.

The differential equation of motion of damped harmonic oscillator is

$$\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + \lambda^2 x = f(t) \quad \dots\dots(1)$$

$$f(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\omega t}\, F(\omega)d\omega$$

&

$$F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\omega t}\, f(t)dt$$

$$F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\omega t}\left(\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + \lambda^2 x(t)\right)dt$$

$$F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{d^2x}{dt^2}e^{-i\omega t}dt + \frac{2k}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{dx}{dt}e^{-i\omega t}dt + \frac{\lambda^2}{\sqrt{2\pi}}\int_{-\infty}^{\infty} x(t)e^{-i\omega t}dt$$

$$F(\omega) = X_2(\omega) + 2kX_1(\omega) + \lambda^2 X(\omega)$$

We Know, $X_n(\omega) = (i\omega)^n X(\omega)$

$$\therefore\ F(\omega) = (i\omega)^2 X(\omega) + 2k(i\omega)^1 X(\omega) + \lambda^2 X(\omega)$$

$$F(\omega) = -\omega^2 X(\omega) + 2ik\omega X(\omega) + \lambda^2 X(\omega)$$

$$F(\omega) = (\lambda^2 - \omega^2 + 2ik\omega)X(\omega)$$

Or $X(\omega) = \frac{F(\omega)}{(\lambda^2-\omega^2+2ik\omega)} \quad \dots\dots\dots(1)$

Where $X(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\omega t}\, x(t)dt$

Taking Fourier inverse transform, The solution for eq^n (1) is given by

$$x(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\omega t}\, X(\omega)d\omega$$

$$x(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{F(\omega)e^{i\omega t}}{(\lambda^2 - \omega^2 + 2ik\omega)}d\omega$$

3) Solution for boundary value problems

The Fourier transform may be applied to solve certain boundary problems like one dimensional heat flow, one dimensional heat equation, etc.

Ex. Solve $\frac{\delta u(x,t)}{\delta t} = \frac{\delta^2(x,t)}{\delta x^2}$ $, > 0$, $t > 0$; subject to conditions $i)\ \ u(0,t)\quad ii)\ u(x,0) = \begin{cases} 1; & 0 < x < 1 \\ 0; & x \geq 1 \end{cases}$

$$iii)\ u(x,t)\ is\ bounded$$

Solution : - The given differential equation is

$$\frac{\delta u}{\delta t} = \frac{\delta^2 x}{\delta x^2}$$

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Taking Fourier sine transform on both side

$$\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\delta u}{\delta t}\,\sin\omega x\;dx = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\delta^2 x}{\delta x^2}\,\sin\omega x\;dx$$

$$\frac{\delta}{\delta t}\sqrt{\frac{2}{\pi}}\int_0^\infty u\;\sin\omega x\;dx$$

$$= \sqrt{\frac{2}{\pi}}\left[\left\{\frac{\delta u}{\delta x}\sin\omega x\right\}_0^\infty - \omega\int_0^\infty \frac{\delta u}{\delta x}\,\cos\omega x\;dx\right]$$

As $x \to \infty; \frac{\delta u}{\delta x} \to 0$ and $x \to 0$; $\sin\omega x \to 0$

Hence$\left\{\frac{\delta u}{\delta x}\sin\omega x\right\}_0^\infty = 0$ &

$$\sqrt{\frac{2}{\pi}}\int_0^\infty u\;\sin\omega x\;dx = u_s \qquad \dots\dots(1)$$

$$\therefore \frac{\delta u_s}{\delta t} = -\sqrt{\frac{2}{\pi}}\,\omega\int_0^\infty \frac{\delta u}{\delta x}\,\cos\omega x\;dx$$

$$\frac{\delta u_s}{\delta t} = \sqrt{\frac{2}{\pi}}\,\omega\left[\{u\cos\omega x\}_0^\infty + \omega\int_0^\infty u\sin\omega x\;dx\right]$$

For $u \to 0\;\;as\;x \to \infty$

$$\frac{\delta u_s}{\delta t} = \sqrt{\frac{2}{\pi}}\,\omega\left[\{u(0,t)\}_0^\infty + \omega\int_0^\infty u\sin\omega x\;dx\right]$$

Using condition (i)

$$\frac{\delta u_s}{\delta t} = -\omega^2 u_s \dots\dots\dots..(2)$$

After rearranging

$$\frac{\delta u_s}{u_s} = -\omega^2\,\delta t$$

Integrating, we get

$\log u_s = -\omega^2 t + \log A$ (A is constant)

Taking antilog on both side

$$u_s = Ae^{-\omega^2 t}$$

At t = 0,

$$u_s(\omega,0) = A \dots\dots..(3)$$

Using eq^n (1), we have

$$u_s(\omega,0) = A = \sqrt{\frac{2}{\pi}}\int_0^\infty u(x,0)\;\sin\omega x\;dx$$

$$= \sqrt{\frac{2}{\pi}}\left[\int_0^1 1\;\sin\omega x\;dx + \int_1^\infty 0\;\sin\omega x\;dx\right]$$

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$$A = \sqrt{\frac{2}{\pi}}\ \left[\frac{\cos \omega x}{\omega}\right]_0^1 + 0$$

$$A = -\sqrt{\frac{2}{\pi}}\ \frac{1-\cos\omega}{\omega}$$

Applying Inverse Fourier sine transform, we get

$$u(x,t) = \sqrt{\frac{2}{\pi}} \int_0^\infty u_s\ \sin\omega x\ d\omega$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{1-\cos\omega}{\omega}\ \sin\omega x\ d\omega$$

This is the required solution.

Note : The result of the following definite integrals are,

1) $\int_0^\infty e^{-t^2} dt = \frac{\sqrt{\pi}}{2}$

2) $\tau(n) = a^n \int_0^\infty e^{-at} t^{n-1} dt$

3) $\int_0^\infty e^{-at} \cos bt\, dt = \frac{a}{a^2+b^2}$

4) $\int_0^\infty \frac{\sin at}{t} dt = \frac{\pi}{2}$ where $a > 0$

5) $\int_0^\infty e^{-at} \sin(bt) dt = \frac{b}{a^2+b^2}$

6) $\int e^{at} \cos(bt) dt = \frac{e^{at}}{(a^2+b^2)}[a\cos(bt) + b\sin(bt)]$

7) $\int e^{at} \sin(bt) dt = \frac{e^{at}}{(a^2+b^2)}[a\sin(bt) - b\sin(bt)]$

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